Common Issues in HW:

① 2.3.3.

$$x_n \le y_n \Rightarrow \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$$

$$y_n \le z_n \Rightarrow \lim_{n\to\infty} y_n \le \lim_{n\to\infty} z_n \quad \text{by Order Limit Theorem.}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l \Rightarrow \lim_{n\to\infty} y_n = l$$

What was wrong?

②. 2.3.10.(a)

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \quad \text{by algebraic limit theorem.}$$

So $\lim_{n\to\infty}(a_n - b_n) = 0 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

What was wrong?

Algebraic Limit Thm and Order Limit Thm requires a priori existence of certain limits

Leftovers in Chap 2 :

① Double Summation of Infinite Series.

Thm : Let $\{a_{ij} : i, j \in \mathbb{Z}_+\}$ be a doubly indexed array of real numbers. If $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ converges, then any rearrangement of the order of summation yields the same limit.

② Cauchy product of series

$$\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right) = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots$$

$$= \sum_{k=2}^{\infty}\sum_{i=1}^{k} a_i b_{k-i}$$

Thm : If $\sum_{i=1}^{\infty} a_i$ converges absolutely to A

$\sum_{j=1}^{\infty} b_j$ converges absolutely to B

then $\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right)$ converges absolutely to AB.

Extremely useful in several complex variable functions

③ Subsequences of bounded sequence.

Let (a_n) be a bounded sequence.

Define $\overline{a_n} = \sup\{a_k : k \geq n\}$, $\underline{a_n} = \inf\{a_k : k \geq n\}$

(1) $\underline{a_n} \leq a_n \leq \overline{a_n}$

(2) $\overline{a_n} \searrow$, $\underline{a_n} \nearrow$

(3) $\lim_{n\to\infty} \overline{a_n}$, $\lim_{n\to\infty} \underline{a_n}$ exists. (as long as (a_n) is bounded)

Define $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \overline{a_n}$,

$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \underline{a_n}$

<u>Thm</u>: TFAE (The following are equivalent)

(a) (a_n) converges

(b) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$

(c) $\lim_{n\to\infty} (\overline{a_n} - \underline{a_n}) = 0$

<u>Pf</u>: (a) $\Rightarrow$ (b)

$a_n \to A \Rightarrow \forall \varepsilon > 0, \exists N > 0, \forall n > N, A - \varepsilon < a_n < A + \varepsilon$

$\Rightarrow A - \varepsilon \leq \inf\{a_k : k \geq n\} = \underline{a_n} \leq \overline{a_n} = \sup\{a_k : k \geq n\} \leq A + \varepsilon$

$\Rightarrow$ both $\underline{a_n}$ and $\overline{a_n}$ converge to A.

(b) $\Rightarrow$ (a)

Follows from squeeze lemma.

(b) $\Leftrightarrow$ (c) obviously.

<u>Prop</u>: $(a_n), (b_n)$ bounded sequences.

(a) If $\exists N_0, \forall n > N_0, a_n \geq b_n$, then

$\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$, $\limsup_{n\to\infty} a_n \geq \limsup_{n\to\infty} b_n$

(b) $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

<u>Pf</u>: (a) If $n > N_0$, then $\underline{b_n} \leq b_k \leq a_k, \forall k \geq n$

$\Rightarrow \underline{b_n} \leq \inf\{a_k : k \geq n\} = \underline{a_n}, \forall n > N_0$

(Order limit thm) $\Rightarrow \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty} a_n$.

Similarly prove $\liminf_{n\to\infty} a_n \geq \liminf_{n\to\infty} b_n$

(b) $a_n + b_n \leq \overline{a_n} + \overline{b_n}$, $\forall n$

$\Rightarrow \limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} (\overline{a_n} + \overline{b_n})$ (from (a)).

$\overline{a_n} + \overline{b_n} \searrow$ and bounded below $\Rightarrow \lim_{n\to\infty} (\overline{a_n} + \overline{b_n})$ exists

$\Rightarrow \limsup_{n\to\infty} (\overline{a_n} + \overline{b_n}) = \lim_{n\to\infty} (\overline{a_n} + \overline{b_n})$ (by Thm)

$= \lim_{n\to\infty} \overline{a_n} + \lim_{n\to\infty} \overline{b_n}$ (Alg. Limit. Thm)

$= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

<u>Example</u>: Find $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ for

(i) $a_n = 1 + (-1)^n$, (ii) $b_n = \frac{1}{\sqrt{n}} + \frac{(-1)^n}{n}$

<u>Example</u>: Find $\limsup_{n\to\infty} (-1)^n$, $\limsup_{n\to\infty} (-1)^{n+1}$ and $\limsup_{n\to\infty} \left((-1)^n + (-1)^{n+1}\right)$.

<u>Thm</u>: Let (a_n) be a bounded sequence. If (a_{k_n}) is a convergent subsequence, then $\lim_{n\to\infty} a_{k_n} \in [\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n]$

<u>Pf</u>: $\underline{a_{k_n}} \leq a_{k_n} \leq \overline{a_{k_n}}$

From the conclusion of 2.5.5,

the subsequence $(\overline{a_{k_n}})$ of $(\overline{a_n})$ converges to $\limsup_{n\to\infty} a_n$

the subsequence $(\underline{a_{k_n}})$ of $(\underline{a_n})$ converges to $\liminf_{n\to\infty} a_n$

Order Limit Thm $\Rightarrow \liminf_{n\to\infty} a_n \leq \lim_{n\to\infty} a_{k_n} \leq \limsup_{n\to\infty} a_n$.

Remark: If we set $b_n = a_{k_n}$, then $\overline{b_n} \neq \overline{a_{k_n}}$. Why?

Thm: Let (a_n) be a bounded sequence, then there exists subsequences (a_{k_n}), (a_{ℓ_n}) s.t. $(a_{k_n}) \to \limsup_{n\to\infty} a_n$, $(a_{\ell_n}) \to \liminf_{n\to\infty} a_n$.

Pf: Since $\overline{a_n} = \sup\{a_k : k \geq n\}$, for $\varepsilon = \frac{1}{n}$, $\exists\, k_n \geq n$, $\overline{a_n} - \frac{1}{n} < a_{k_n} < \overline{a_n}$

Notice that $\lim_{n\to\infty}\left(\overline{a_n} - \frac{1}{n}\right) = \lim_{n\to\infty} \overline{a_n} = \limsup_{n\to\infty} a_n$.

hence by squeeze lemma, $\lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n$.

Similarly one can find $(a_{\ell_n}) \to \liminf_{n\to\infty} a_n$

Remark: If we set $L(a_n) = \{$ limits of all convergent subsequences $\}$,

then $\limsup_{n\to\infty} a_n = \sup L(a_n)$, $\liminf_{n\to\infty} a_n = \inf L(a_n)$.

HW to be graded: 2.4.7, 2.5.1*, 2.5.5*, 2.6.2, 2.7.1(b).

My sol'n to 2.5.1.

(a) Impossible. A bounded subsequence contains a convergent subsequence by B-W.

(b) $a_n = \begin{cases} \frac{1}{n} & n \text{ even} \\ 1+\frac{1}{n} & n \text{ odd} \end{cases}$

(c) Since $\mathbb{N} \times \mathbb{N}$ is countable, let $\alpha: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $n \mapsto (\alpha_1(n), \alpha_2(n))$ be the bijection and define $a_n = \left(\frac{1}{\alpha_1(n)} + \frac{1}{\alpha_2(n)}\right)$

$\forall k \in \mathbb{N}$, $\forall \ell \in \mathbb{N}$, $\exists n_{k,\ell}$. s.t. $\alpha_1(n_{k,\ell}) = \frac{1}{k}$, $\alpha_2(n_{k,\ell}) = \frac{1}{\ell}$

So for fixed $k = 1, 2, \cdots$, the subsequence $a_{m_\ell} = a_{n_{k,\ell}}$ converges to $\frac{1}{k}$ as $\ell \to \infty$.

(d). Impossible. If there exists a subsequence converging to $\frac{1}{k}$, then for $\varepsilon = \frac{1}{4^k}$, $\exists a_{n_k}$, $|a_{n_k} - \frac{1}{k}| < \varepsilon = \frac{1}{4^k}$, $k = 1, 2, \cdots$

The so-chosen subsequence $(a_{n_1}, a_{n_2}, \cdots, a_{n_k}, \cdots)$ satisfies

$$\frac{1}{k} - \frac{1}{4^k} < a_{n_k} < \frac{1}{k} + \frac{1}{4^k}$$

$\lim_{k\to\infty}\left(\frac{1}{k} - \frac{1}{4^k}\right) = \lim_{k\to\infty}\left(\frac{1}{k} + \frac{1}{4^k}\right) = 0$. Squeeze lemma $\Rightarrow \lim_{k\to\infty} a_{n_k} = 0$

So there exists a subsequence that converges to 0.

My solution to 2.5.5. Assuming (a_n) is bounded, then

$(a_n) \to a \iff \forall$ convergent subsequence $(a_{n_k}) \to a$

$\Rightarrow$: $(a_n) \to a \Rightarrow \forall \varepsilon > 0, \exists N > 0, \forall n > N, |a_n - a| < \varepsilon$.

Then for every subsequence (a_{n_k}), by $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$,

$\exists K > 0$, s.t. $\forall k > K$, $n_k > N$.

$\Rightarrow \forall k > K$, $|a_{n_k} - a| < \varepsilon$. $\Rightarrow \lim_{k\to\infty} a_{n_k} = a$.

$\Leftarrow$: We prove the contrapositive statement:

$(a_n) \not\to a \Longrightarrow \exists$ convergent subsequence (a_{n_k}), $(a_{n_k}) \not\to a$.

$(a_n) \not\to a \Rightarrow \exists\, \varepsilon > 0,\ \forall N > 0,\ \exists\, n > N,\ |a_n - a| \geq \varepsilon.$

Fixing this ε, for $N = 1$, $\exists\, n_1 > 1$, $|a_{n_1} - a| \geq \varepsilon$

$N = n_1$, $\exists\, n_2 > n_1$, $|a_{n_2} - a| \geq \varepsilon$

$\vdots \qquad \vdots \qquad \vdots$

$N = n_k$, $\exists\, n_{k+1} > n_k$, $|a_{n_{k+1}} - a| \geq \varepsilon$

$\vdots \qquad \vdots$

Therefore we have chosen a subsequence (a_{n_k}) that does not converge to a. By (a_n) bdd, (a_{n_k}) bdd $\Rightarrow \exists$ convergent subseq. of (a_{n_k}) (i.e. of (a_n)) that does not converge to a. $\square$.